Speed of explosion for continuous-state nonlinear branching processes with big jumps

Xiaowen Zhou

Concordia University

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Outline of the Talk

Continuous-state branching process (CSBP)

- Continuous-state branching process arises as a time-space scaling limit of the classical discrete-state branching processes.
- It is a nonnegative Markov process X (with no negative jumps) satisfying the additive branching property: the sum of any two independent CSBPs with the same branching mechanism is a CSBP.
- Let X and Y be two independent CSBPs with initial values x and y, respectively. Then

$$\mathbb{E}_{x+y}e^{-\theta X_t} = \mathbb{E}_x e^{-\theta X_t}\mathbb{E}_y e^{-\theta X_t}, \;\; x, y > 0,$$

Laplace transform of CSBP

- Its Laplace transform is determined by

$$\mathbb{E}_x e^{-\theta X_t} = e^{-x u_t(\theta)}$$

where function $u_t(\theta)$ satisfies

$$\frac{\partial u_t(\theta)}{\partial t} + \psi(u_t(\theta)) = 0$$

with $u_0(\theta) = \theta$ and branching mechanism

$$\psi(\lambda) = b\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x 1_{x \le 1})\nu(\mathrm{d}x)$$

for $\sigma > 0$, $b \in \mathbb{R}$ and measure ν satisfying $\int_0^\infty (1 \wedge z^2)\nu(\mathrm{d}z) < \infty$.

- X is critical, subcritical, supercritical if $\mathbb{E}X_t$ is constant, decreasing or increasing, respectively.

CSBP as solution to SDE

The continuous-state branching process is the unique nonnegative solution to

$$X_t = X_0 + bX_t + \sigma \int_0^t \int_0^{X_{t-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^{X_{s-}} \int_0^1 z\tilde{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z) + \int_0^t \int_0^{X_{s-}} \int_1^\infty zN(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z)$$

where $W(\mathrm{d}s, \mathrm{d}u)$ is a Gaussian white noise on $(0, \infty)^2$ and $\tilde{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z)$ is an independent compensated spectrally positive Poisson random measure on $(0, \infty)^3$ with intensity $\mathrm{d}s\mathrm{d}u\nu(\mathrm{d}z)$.

Why CSBP?

- The relation between Bienaymé-Galton-Watson branching process and CSBP is like that between Markov chain and Brownian motion.
- On one hand, CSBPs keep the key common features and ignore the minor differences between the discrete-state branching processes.
- On the other hand, by considering the continuous-state processes we can apply Lévy process and SDE theories.

Why is CSBP spectrally positive?

- The continuous-time discrete-state branching process has negative jumps of size 1, which disappears in the scaling limit.

Spectrally positive Lévy process (SPLP)

- A spectrally positive Lévy process ξ has stationary independent increments and no negative jumps.
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$$\mathbb{E}e^{-\lambda(\xi_t-\xi_0)} = e^{t\psi(\lambda)},$$
for $\lambda, t \geq 0$, where the Laplace exponent for $-\xi$ is
$$\psi(\lambda) = b\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty \left(e^{-\lambda z} - 1 + \lambda z 1_{z\leq 1}\right)\nu(dz),$$
for $b \in \mathbb{R}$ and $\sigma \geq 0$. We assume that the σ-finite Lévy measure ν on $(0, \infty)$ satisfies $\int_0^\infty (1 \wedge z^2)\nu(dz) < \infty$.
- The scale function W of the process $-\xi$ is defined as the function with Laplace transform on $[0, \infty)$ given by
$$\int_0^\infty e^{-\lambda z} W(z)dz = \frac{1}{\psi(\lambda)} \quad \text{for } \lambda > \Phi(0) := \psi^{-1}(0).$$

Lamperti transform

- CSBP is associated with SPLP via the Lamperti time change.
- Write ξ for a spectrally positive Lévy process and $\tau_0^- := \inf\{t : \xi_t = 0\}$ for its first time of reaching 0. Write

$$\eta(t) := \int_0^{t\wedge\tau_0^-} \frac{1}{\xi_s} ds \quad \text{and} \quad \eta^{-1}(t) := \inf\{s \geq 0 : \eta(s) > t\}.$$

  Then process $X_t := \xi_{\eta^{-1}(t)\wedge\tau_0^-}$ is the CSBP.
- Loosely speaking, process X_t runs through the sample path of ξ_t at a different speed. X_t speeds up when it takes large values and slows down when it takes small values.
- The Lamperti transform changes the times of the jumps but keep the same starting positions and sizes of the jumps.

A class of continuous-state nonlinear branching processes

- We can generalize the Lamperti transform to introduce nonlinear CSBPs. Let R be a positive continuous function on $(0, \infty)$ satisfying $\inf_{x>\epsilon} R(x) > 0, \forall \epsilon > 0$.
- Define

$$\eta(t) := \int_0^{t\wedge\tau_0^-} \frac{1}{R(\xi_s)} ds, \quad \eta^{-1}(t) := \inf\{s < \tau_0^- : \eta(s) > t\},$$

Let

$$X_t := \begin{cases} \xi_{\eta^{-1}(t)\wedge\tau_0^-}, & t < \eta(\tau_0^-), \\ \xi_{\eta(\tau_0^-)-}, & t \geq \eta(\tau_0^-), \end{cases}$$

be a continuous-state nonlinear branching process with branching rate function $R(\cdot)$.

- Such nonlinear CSBPs were first introduced in Li (2019).

Different ways of characterizing nonlinear CSBP

- Process X has a generator L on $C^2(0,\infty)$ such that

$$\begin{aligned} Lf(x) &:= R(x)L^*f(x) \\ &= R(x)\Big(bf'(x) + \frac{\gamma}{2}f''(x) \\ &\qquad + \int_0^\infty (f(x+u) - f(x) - uf'(x)1_{z\le 1})\nu(du)\Big). \end{aligned}$$

- It is also the unique nonnegative solution (stopped at 0) to

$$\begin{aligned} X_t &= X_0 + \int_0^t bR(X_s)\mathrm{d}s + \int_0^t \sqrt{\gamma R(X_s)}\mathrm{d}B_s \\ &+ \int_0^t \int_0^{R(X_{s-})} \int_0^1 z\tilde{N}(\mathrm{d}s,\mathrm{d}u,\mathrm{d}z) + \int_0^t \int_0^{R(X_{s-})} \int_1^\infty zN(\mathrm{d}s,\mathrm{d}u,\mathrm{d}z \end{aligned}$$

Why nonlinear branching mechanism?

- The nonlinear CSBP allows the branching rate to depend on the current population size and can be a more flexible population model.
- The nonlinear branching mechanism allows the process to have richer asymptotic behaviors.
  - The process comes down from infinity if "starting from ∞", it becomes finite at any positive time, which can not happen to (linear) CSBPs. The speed of CDI for nonlinear CSBP is studied in Foucart, Li P.S. and Z. (2021).
  - Explosion occurs if the process approaches to $+\infty$ in finite time, which can not happen to CSBP if its large jumps have finite mean. The explosion behaviours of nonlinear CSBP is studied in Li B. and Z. (2021).
  - The process becomes extinguishing if it converges to 0 but never reaches 0. The extinguishing behaviours of nonlinear CSBP is studied in Li J., Tang and Z. (2022).

Difficulty with working on nonlinear CSBP

- The additive branching property and the associated classical techniques fail in handling nonlinear CSBP.
- We need to introduce new techniques such as
  - Lévy process techniques,
  - SDE techniques.

How to reach infinity in finite time (explosion)?

- Let $T_x^+ := \inf\{t : X_t \geq x\}$ and $T_\infty^+ := \lim_{x\to\infty} T_x^+$ be the explosion time if $T_\infty^+ < \infty$. Observe that

$$T_\infty^+ = \begin{cases} \int_0^\infty \frac{1}{R(\xi_s)} \mathrm{d}s & \text{if } \tau_0^- = \infty, \\ \infty & \text{if } \tau_0^- < \infty. \end{cases}$$

- For explosion to happen to X, we need
  - $\xi_t \to \infty$ as $t \to \infty$ (X is supercritical);
  - either $R(x) \to \infty$ fast enough as $x \to \infty$ or ξ_t has large enough jumps.
- Suppose that $R(x)$ goes to ∞ fast enough as $x \to \infty$. If X has reached a high level x, then due to the time change, it will speed up on its way to ∞ and can even reach ∞ within finite time, i.e. explosion occurs.
- We are not aware of previous results on speed of explosion for general Markov processes. Some related results for CSBPs can be found in Boinghoff and Hutzenthaler (2012) and Palau and

The goal and the approach

- There are different ways of characterizing speed of explosion in terms of "how fast X_s grows as $s \to T_\infty^+$".
    - Given $T_\infty^+ < \infty$, $X(T_\infty^+ - t) \to \infty$ as $t \to 0+$. We want to know how fast $X(T_\infty^+ - t)$ increases as $t \to 0+$.
    - A closely related problem is how fast the upcrossing time T_x^+ approaches to T_∞^+ as $x \to \infty$.
- The speed of explosion is not known even for CSBP.
- We use some ideas from Foucart et al (2021) for coming down from infinity.
- A main difficulty comes from overshoot when X first upcrosses a level. Such a difficulty does not appear in studying CDI.

Nonlinear CSBP with small jumps

- (Explosion criterion) For $\gamma := \mathbb{E}(\xi_1 - \xi_0) \in (0, \infty)$, explosion occurs for X if and only if $\int^\infty \frac{1}{R(y)} \mathrm{d}y < \infty$; see Döring and Kyprianou (2016).
- By the above integral test, CSBP with $R(x) = x$ and $\gamma < \infty$ can not explode.
- Under the above conditions, the overshoot of X when upcrossing a level has an asymptotic stationary distribution,

$$\mathbb{P}(X(T_y^+) - y \in \mathrm{d}z) = \mathbb{P}(\xi(\tau_y^+) - y \in \mathrm{d}z) \Rightarrow \rho(\mathrm{d}z) \quad \text{as } y \to \infty,$$

where $\int_{0-}^\infty e^{-sz} \rho(\mathrm{d}z) = \frac{\Phi(0)\psi(s)}{\gamma s(s - \Phi(0))}$.

Define $\mathbb{P}_1^{\uparrow}(\cdot) := \mathbb{P}_1\big(\cdot \,\big|\, T_\infty^+ < \infty\big)$. Let $\varphi(x) := \frac{1}{\gamma}\int_x^\infty \frac{dy}{R(y)}$.
Function g is regularly varying with index γ if $g(\lambda x)/g(x) \sim \lambda^\gamma$.
Write $\mathcal{R}_\gamma$ for the set of regular varying functions with index γ.

Theorem

(Li and Z. EJP 2021) If $0 < \gamma < \infty$ and $R \in \mathcal{R}_\beta$ at ∞ for $\beta > 1$, we have

$$\frac{T_\infty^+ - T_x^+}{\varphi(x)}\Big|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \Rightarrow 1 \quad \textit{in probability as } x \to \infty;$$

$$\frac{X(T_\infty^+ - t)}{\varphi^{-1}(t)}\Big|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \Rightarrow 1 \quad \textit{in probability as } t \to 0+.$$

The nonlinear CSBP explodes along a deterministic curve.

The speed of explosion for CSBP was left open.

Nonlinear CSBPs with large jumps

- In the above results on the speed of explosion, a key assumption is that the big jumps have a finite first moment, which guarantees stationary overshoot.
- Unfortunately, these results can not be applied to (linear) CSBP since for explosion to happen to CSBP, the associated SPLP is necessary to have big jumps of infinite mean, i.e. $\int_1^\infty x\nu(dx) = \infty$.
- We recently consider the speed of explosion for nonlinear CSBP with big jumps.
- (Explosion criterion) If $\xi_t \to \infty$ and $R(x) \uparrow \infty$, then $\mathbb{P}_x(T_\infty^+ < \infty) > 0$ if and only if for all $x > a > 0$,

$$\mathbb{E}_x[\eta(\infty); \tau_a^- = \infty]$$
$$= \int_a^\infty \frac{1}{R(y)} \left(e^{\Phi(0)(a-x)} W(y-a) - W(y-x)\right) dy < \infty.$$

Assumptions for nonlinear CSBPs with large jumps

- To handle the large overshoot, we want to show the rescaled overshoot converges for which we need to impose some scaling properties on rate function R and Laplace exponent ψ.
- In the rest of the talk, we assume that $\psi'(0) = -\infty$, $R \in \mathcal{R}_\beta$ at ∞ and $-\psi \in \mathcal{R}_\alpha$ at 0 for some $\beta \geq \alpha$ and $\alpha \in (0, 1]$.
- We first consider first the case $\beta > \alpha \geq 0$ in which explosion happens.
- Define for large x

$$\varphi(x) := \int_x^\infty \frac{-1}{yR(y)\psi(1/y)}\, dy \in \mathcal{R}_{\alpha-\beta}, \tag{1}$$

$$\varphi^{-1}(t) := \inf\{r > 0, \varphi(r) = t\}, \quad t > 0.$$

Then

$$\varphi(x) \sim \frac{1}{\beta - \alpha} \times \frac{-1}{R(x)\psi(1/x)} \in \mathcal{R}_{\alpha-\beta} \text{ at } \infty.$$

Theorem

(Foucart, Li and Z. 2023+) For $\beta > \alpha > 0$ *we have*

$$\frac{T_\infty^+ - T_x^+}{\varphi(x)}\Big|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \Rightarrow \chi_\alpha^{\alpha-\beta} \times \varrho$$

where ϱ *has the same distribution as* $\int_0^\infty \eta_t^{-\beta} dt$ *for* α*-stable subordinator* η_t*,*

$$\mathbb{E}[e^{\theta\varrho}] = 1 + \sum_{n\geq 1} \frac{\theta^n}{n!}\Big(\prod_{k=1}^n \frac{\Gamma(k(\beta-\alpha)+1)}{\Gamma(k(\beta-\alpha)+\alpha)}\Big) \quad \text{for all } \theta \in \mathbb{R}$$

and where $\chi_\alpha \geq 1$ *is a random variable independent of* ϱ *with*

$$\mathbb{P}(\chi_\alpha > t) = \frac{\sin \alpha\pi}{\pi} \int_0^1 u^{\alpha-1}(t-u)^{-\alpha}\, du \quad \text{for } t \geq 1.$$

In particular, $\chi_\alpha^{\alpha-\beta} \times \varrho = 1$ *if* $\alpha = 1$*. Recall* $\beta = 1$ *for CSBP.*

Outline of the approach

Given $T_\infty^+ < \infty$, we first express $T_\infty^+ - T_x^+$ in terms of SPLP ξ, and then work on the weighted occupation time for ξ. Applying the strong Markov property at τ_x^+,

$$
\begin{aligned}
\frac{T_\infty^+ - T_x^+}{\varphi(x)} &= \frac{1}{\varphi(x)} \int_{\tau_x^+}^{\infty} \frac{1}{R(\xi(t))}\, dt \\
&= \frac{\eta(\infty) \circ \theta_{\tau_x^+}}{\varphi(x)} \\
&= \frac{\varphi(\xi(\tau_x^+))}{\varphi(x)} \times \frac{\eta(\infty) \circ \theta_{\tau_x^+}}{\varphi(\xi(\tau_x^+))},
\end{aligned}
$$

where θ_t denotes the shift operator, $\frac{\varphi(\xi(\tau_x^+))}{\varphi(x)}$ concerns the overshoot and $\frac{\eta(\infty)\circ\theta_{\tau_x^+}}{\varphi(\xi(\tau_x^+))}$ concerns the residual explosion time after the first passage time τ_x^+.

We first show that the overshoot after re-scaling converges.

Proposition

Suppose that $-\psi \in \mathcal{R}_\alpha$ *at* 0 *for some* $\alpha \in (0, 1]$. *Then*

$$x^{-1}\xi(\tau_x^+) \Rightarrow \chi_\alpha \quad \text{as} \quad x \to \infty.$$

If $\alpha = 1$, *then* $x^{-1}\xi(\tau_x^+) \Rightarrow 1$.

- Observe that the overshoot of a nonlinear CSBP is the same as that of the associated SPLP, which is the same as that of the ladder height process for SPLP.
- Since the ladder height process is a subordinator, we can then apply Bertoin's result on stable like subordinators.

For the limit of $\frac{\eta(\infty)\circ\theta_{\tau_x^+}}{\varphi(\xi(\tau_x^+))}$ we prove the following result.

Proposition

Suppose $R \in \mathcal{R}_\beta$ *at* ∞ *and* $-\psi \in \mathcal{R}_\alpha$ *at* 0 *with* $\beta > \alpha \geq 0$, *we have*

$$\left.\frac{T_\infty^+}{\varphi(x)}\right|_{\mathbb{P}_x(\cdot|T_\infty^+<\infty)} \Rightarrow \varrho \quad \text{as } x \to \infty.$$

Proof of the main theorem

By the previous Propositions,

$$\frac{\varphi(\xi(\tau_x^+))}{\varphi(x)} \sim \varphi\left(\frac{\xi(\tau_x^+)}{x}\right) \Rightarrow \chi_\alpha^{\alpha-\beta} \quad \text{as } x \to \infty$$

and

$$\frac{\eta(\infty) \circ \theta_{\tau_x^+}}{\varphi(\xi(\tau_x^+))}\Big|_{\mathbb{P}_x(\cdot | T_\infty^+ < \infty)} \Rightarrow \varrho \quad \text{as } x \to \infty.$$

Then

$$\frac{T_\infty^+ - T_x^+}{\varphi(x)}\Big|_{\mathbb{P}_1(\cdot | T_\infty^+ < \infty)} \Rightarrow \chi_\alpha^{\alpha-\beta} \times \varrho.$$

Theorem

Suppose $\beta > \alpha > 0$, *we have*

$$\frac{X(T_\infty^+ - t)}{\varphi^{-1}(t)}\Big|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \Rightarrow \frac{1}{\chi_\alpha} \times \varrho^{\frac{1}{\beta-\alpha}}.$$

In particular, if $\alpha = 1$, *then Suppose* $\beta > \alpha > 0$, *we have*

$$\frac{X(T_\infty^+ - t)}{\varphi^{-1}(t)}\Big|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \Rightarrow 1.$$

Speed of explosion in the critical case $1 > \beta = \alpha > 0$

Theorem

Suppose $R \in \mathcal{R}_\alpha$ *at* ∞ *and* $-\psi \in \mathcal{R}_\alpha$ *at* 0 *for* $\alpha \in (0,1)$, *then*

$$\mathbb{P}_1\big(T_\infty^+ < \infty\big) > 0 \quad \text{if and only if} \quad \int^\infty \frac{-1}{R(y)\psi(1/y)y}\,dy < \infty.$$

In this case,

$$\frac{T_\infty^+ - T_x^+}{\varphi(x)}\bigg|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \Rightarrow \Gamma(\alpha) \quad \text{in probability.}$$

Thank you for your attention!